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Note

Kernels in graphs with a clique-cutset

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Abstract

We consider graphs that have a clique-cutset, and we show that this property preserves the existence of a kernel in a certain sense.

We consider finite directed graphs that do not have multiple arcs or loops, but there may be symmetric arcs between some pairs of vertices. Let $G = (V, A)$ be a directed graph, where V is the vertex-set and A is the arc-set. A *kernel* of G is a subset K of vertices such that (a) every vertex of $G - K$ has a successor in K (the kernel is *absorbant*) and (b) there is no arc between any two vertices of the kernel (the kernel is *stable*). Not every directed graph admits a kernel; in fact it was shown by Chvátal [2] that deciding if a graph possesses a kernel is an NP-complete problem; this result was strengthened by Fraenkel [3]. On the other hand, many conditions are known that imply the existence of a kernel, see e.g. [1]. Such conditions are usually hereditary, and so they also imply the existence of a kernel for every induced subgraph. A directed graph such that every induced subgraph has a kernel is called *kernel-perfect*. By similarity with other topics in graph theory, it is interesting to know what kind of graph constructions preserve kernel-perfectness. Here we consider the *clique cutset*. A clique-cutset is a subset C of vertices such that C is a clique and $G - C$ is disconnected. For each connected component A of $G - G$ we call the subgraph induced by $A \cup C$ a *piece* of G with respect to C. The subgraph of G induced by a subset S will be denoted by $G[S]$. The main result of this paper is the following:

Theorem 1. *Let G be a directed graph which admits a clique-cutset C. Then G is kernel-perfect if and only if every piece of G with respect to C is kernel-perfect.*

This theorem should be viewed in parallel with some similar classical results in graph theory, in particular: *if a graph admits a clique-cutset C, then it is*

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vertex-k-colorable if and only if every piece of G with respect to C is vertex-k-colorable (see [1]). See also [4] for some application of this theorem.

Proof of the theorem. It is clear that if G is kernel-perfect all pieces are kernel-perfect, since they are induced subgraphs of G.

Now consider a graph G with a clique-cutset C such that all pieces of G with respect to C are kernel-perfect, and let us prove by induction on $|V|$ that G is kernel-perfect. The fact is trivial for $|V| \leqslant 3$. For $|V| \geqslant 4$, we need only prove that G has a kernel, since every proper induced subgraph H of G either is a subgraph of some piece of G with respect to C or has a clique-cutset $C \cap H$ whose pieces are smaller induced subgraphs of the pieces of G, and thus H has a kernel by the induction hypothesis.

Let A be a connected component of $G - C$, and $B = G - C - A$ (the subgraph $G[B]$ need not be connected, but this will not be necessary in the proof).

$$E_1 = \{x \in C \mid x \text{ belongs to a kernel of } G[A \cup C]\};$$

$$F_1 = \{x \in C \mid x \text{ belongs to a kernel of } G[B \cup C]\}.$$

These sets may be empty. In general, suppose we have defined non-empty sets $E_1, \dots, E_k$ and consider the set

$$E_{k+1} = \{x \in C \mid x \text{ belongs to a kernel of } G[A \cup C - (E_1 \cup \cdots \cup E_k)]\}.$$

Likewise, suppose we have defined non-empty sets $F_1, \dots, F_q$ ($q \geqslant 1$), and consider the set

$$F_{q+1} = \{x \in C \mid x \text{ belongs to a kernel of } G[B \cup C - (F_1 \cup \cdots \cup F_q)]\}.$$

The sets E_{k+1} and F_{q+1} are well-defined since by the induction hypothesis the proper induced subgraphs $G[A \cup C]$ and $G[B \cup C]$ are kernel-perfect. It may be that when we build the above sets there exist subscripts i, j with

$$E_i \cap F_j \neq \emptyset. \tag{1}$$

If (1) happens, let us take the smallest subscript i_0 for which there exists a subscript j such that $E_{i_0} \cap F_j \neq \emptyset$, and then take the smallest j_0 such that $E_{i_0} \cap F_{j_0} \neq \emptyset$. By this choice, we have

$$E_i \cap F_j = \emptyset \quad \text{for all } i < i_0 \text{ and } j < j_0. \tag{2}$$

Pick a vertex x_0 in $E_{i_0} \cap F_{j_0}$. So $x_0 \in A$ and by the definition of E_{i_0} and F_{j_0} there exists a kernel K_A of the induced subgraph $G[A \cup C - (E_1 \cup \cdots \cup E_{i_0 - 1})]$ and a kernel K_B of the induced subgraph $G[B \cup C - (F_1 \cup \cdots \cup F_{j_0 - 1})]$, with $x_0 \in K_A \cap K_B$. We claim that $K_A \cup K_B$ is a kernel of G. First, it is a stable set. Indeed, since A is a clique, we can write $K_A = (K_A \cap A) \cup \{x_0\}$ and $K_B = (K_B \cap B) \cup \{x_0\}$, so $K_A \cup K_B = (K_A \cap A) \cup (K_B \cap B) \cup \{x_0\}$ where it is clear that this is a stable set, because there is no edge between A and B. Second, $K_A \cup K_B$ is absorbant. Indeed, consider an arbitrary vertex v of $G - (K_A \cup K_B)$, and by symmetry assume $v \in (A \cup C) - K_A$. If v is not in $E_1 \cup \cdots \cup E_{i_0-1}$ then v has a successor in K_A by the definition of K_A. If v is in E_i with $1 \leqslant i < i_0$ then v is in $C - (F_1 \cup \cdots \cup F_{j_0-1})$ because of (2). Hence v has a successor in K_B by the definition of K_B.

Now we may assume that condition (1) never happens. Since A is a finite set, there exist positive integers i_0 and j_0 such that $E_{i_0} = \emptyset$ and $E_i \neq \emptyset$ for all i with $1 \leqslant i \leqslant i_0$, and $F_{j_0} = \emptyset$ and $F_j \neq \emptyset$ for all j with $1 \leqslant j < j_0$. Since (1) never happens, Eq. (2) holds again, with the same notation. Since $G[A \cup C - (E_1 \cup \cdots \cup E_{i_0-1})]$ is kernel-perfect it has a kernel K_A, and the fact that E_{i_0} is empty implies $K_A \subseteq A$. Likewise $G[B \cup C - (F_1 \cup \cdots \cup F_{j_0-1})]$ has a kernel K_B, and $K_B \subseteq B$ because F_{j_0} is empty. We claim that $K_A \cup K_B$ is a kernel of G. It is clear that it is a stable set since $K_A \subseteq A$ and $K_B \subseteq B$. Moreover, it is absorbant. Indeed, let v be an arbitrary vertex in $G - (K_A \cup K_B)$. By symmetry we may assume that v is in $A \cup C$. If v is not in $E_1 \cup \cdots \cup E_{i0-1}$ then v has a successor in K_A by the definition of K_A. If v is in E_i with $1 \leqslant i < i_0$ then (2) implies that v is in $C - (F_1 \cup \cdots \cup F_{j_0-1})$; hence v has a successor in K_B by the definition of K_B. This completes the proof. $\square$

References

[1] C. Berge, Graphs (North-Holland, Amsterdam, 1985).

[2] V. Chvátal, On the computational complexity of finding a kernel, Report CRM300, Centre de Recherches Mathématiques, Université de Montréal, 1973.

[3] A.S. Fraenkel, Planar Kernel and Grundy with $d \leqslant 3$, $d_{\text{out}} \leqslant 2$, $d_{\text{in}} \leqslant 2$, are NP-complete, Discrete Appl. Math. 3 (1981) 257–262.

[4] F. Maffray, Kernels in perfect line-graphs, J. Combin. Theory. Ser. B 55 (1992) 1–8.